2. Differentiation

- Suppose f is a real function and c is a point in its domain. Then, the derivative of f at c is defined by, $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$
- Derivative of a function $f(x)$, denoted by $\frac{d}{dx}(f(x))$ of $f'(x)$, is defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Example:
Find derivative of sin $2x$.

Solution:
Let $f(x) = \sin 2x$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$$
$$= \lim_{h \to 0} \frac{2\cos(2x+h) \cdot \sin h}{h}$$
$$= 2\lim_{h \to 0} \cos(2x+h) \cdot \lim_{h \to 0} \frac{\sin h}{h}$$
$$= 2 \times \cos 2x \times 1$$
$$= 2\cos 2x$$

- For two functions f and g, the rules of algebra of derivatives are as follows:
  - $(f+g)' = f' + g'$
  - $(f-g)' = f' - g'$
  - $(fg)' = f'g'$ [Leibnitz or product rule]
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, where $g \neq 0$ [Quotient rule]
- Every differentiable function is continuous, but the converse is not true.

Example:
$f(x) = |x|$ is continuous at all points on real line, but it is not differentiable at $x = 0$.

Since L.H.S $= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$

R.H.S $= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$

$\therefore$L.H.S $\neq$ R.H.S.

Therefore, $f'(x)$ does not exist at $x = 0$; i.e., f is not differentiable at $x = 0$.

The derivatives of some useful functions are as follows:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

- **Chain rule:** This rule is used to find the derivative of a composite function. Let $f = v \text{ o } u$. Suppose $t = u(x)$; and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

  Similarly, if $f = (w \text{ o } u) \text{ o } v$, and if $t = v(x)$, $s = u(t)$, then $\frac{df}{dx} = \frac{d(w o u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

  Example: Find the derivative of $\sin^2(\log x + \cos^2 x)$.

  Solution:

$$\frac{d}{dx}\left[\sin^2\left(\log x + \cos^2 x\right)\right] = 2\sin\left(\log x + \cos^2 x\right) \times \frac{d}{dx}\left[\sin\left(\log x + \cos^2 x\right)\right]$$
$$= 2\sin\left(\log x + \cos^2 x\right) \cdot \cos\left(\log x + \cos^2 x\right) \times \frac{d}{dx}\left(\log x + \cos^2 x\right)$$
$$= \sin 2\left(\log x + \cos^2 x\right) \cdot \left[\frac{1}{x} + 2\cos x \times \frac{d}{dx}(\cos x)\right]$$
$$= \sin\left(\log x^2 + 2\cos^2 x\right) \times \left(\frac{1}{x} - 2\sin x\cos x\right)$$
$$= \left(\frac{1}{x} - \sin 2x\right)\sin\left(\log x^2 + 2\cos^2 x\right)$$

- Derivative of a function $f(x) = [u(x)]^{v(x)}$ can be calculated by taking logarithm on both the sides, i.e. $\log f(x) = v(x)\log[u(x)]$, and then differentiating both sides with respect to x.

  Example: If $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, find $\frac{dy}{dx}$

  Solution:

  Let If $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^y$

$\therefore \log y = y \log x$

$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}\left[\frac{1}{y} - \log x\right] = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \log x} = \frac{y^2}{x - xy\log x}$

- If the variables x and y are expressed in the form of $x = f(t)$ and $y = g(t)$, then they are said to be in parametric form. In this case, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$

- If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{d^2y}{dx^2}$ or $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

  Here, $f''(x)$ or $\frac{d^2y}{dx^2}$ is called the second order derivative of y with respect to x.